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A NEW FORMULATION OF THE PROBLEM OF THE FLUTTER OF A HOLLOW SHELL†

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A formulation of the problem of the panel flutter of a shell with an arbitrary plan view which is arbitrarily orientated with respect to the flow velocity vector in presented under the assumption that the excess pressure, as viewed from the direction of the flow of gas onto the hollow shell around which the flow occurs, can be determined using the linearized (piston) theory of a supporting surface. The general formulation is made specific using certain examples.

With the rare exception of [5], the objects of investigations on panel flutter [1–5] have been a rectangular plate, a cylindrical panel or a closed cylindrical shell. In these cases, a quite particular formulation of the problem was used which was subject to the condition that the flow velocity vector is parallel to one of the sides of the plate or to the generatrix of the cylindrical shell or panel. The large variety of problems considered was due to the diversity of the boundary conditions and, also, the methods of investigation: rigorous analytical, approximate and numerical methods. The flutter of visco-elastic rectangular plates has been investigated with the same constraints in the formulation in [6–8]. However, the panelling elements of aircraft are either hollow shells with various plan view outlines or circular plates, for example, but not rectangular plates. On the other hand, in many important practical cases, the flow velocity vector is quite arbitrarily orientated with respect to the sides of the plate or panel. The solution of this kind of problem is possible using a general formulation of the problem of panel flutter which has not been given up to now. This deficiency is made good in this paper.

1. Consider a shell in a fixed system of Cartesian coordinates $\{x, y, z\}$. As a geometric surface, the shell (its median surface) is parametrized by the curvilinear coordinates x^1, x^2 and its position in space is determined by the radius vector

$$\mathbf{r}=\{x(x^1,x^2),\quad y(x^1,x^2),\quad z(x^1,x^2)\}$$

The first and second quadratic forms are thereby determined.

$$I_1=g_{ik}dx^idx^k,\quad I_2=b_{ik}dx^idx^k,\quad i,k=1,2$$

The coefficients of the latter are expressed in the well-known manner [9] in terms of the derivative of $\mathbf{r}$ with respect to x^k (we shall subsequently adopt the notation $\mathbf{r}_k=\partial\mathbf{r}/\partial x^k$). The vector of the unit normal to the initial (undeformed) surface of the shell is defined in the following manner

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$$\mathbf{n} = [\mathbf{r}_1 \times \mathbf{r}_2] / |[\mathbf{r}_1 \times \mathbf{r}_2]| \tag{1.1}$$

Let us assume that there is a flow around the external surface of the shell, which is defined by the normal in accordance with (1.1), by a supersonic gas stream with a velocity vector $\mathbf{V} = \{\upsilon_x, \upsilon_y, \upsilon_z\}$. Here, $|\upsilon_z / |\mathbf{V}|| \ll \varepsilon$, where ε is a parameter, which occurs in the estimate of the accuracy when determining the pressure using the law of plane sections [10, 11].

Henceforth we shall consider the flow around a hollow shell so that the condition $|1 - \cos\gamma| \ll 1$ is satisfied, where γ is the angle made by the normal to the surface with the z-axis.

This enables one to use the linearized (piston) theory to determine the pressure on an element of the surface, according to which [11]

$$p - p_0 = \Delta p = \beta \upsilon(t), \quad \beta = \kappa p_0 / C_0 \tag{1.2}$$

Here p_0 and C_0 are the pressure and the velocity of sound in the unperturbed flow and κ is the polytropy index. If the pressure is determined on an element of the panelling which is located behind a shock wave then, generally speaking, it is necessary, instead of p_0, C_0 to take the parameters p_1, C_1, which are defined as averages over the volume enclosed by a right cylinder, the end of which is the shell and the shock wave front.

If the shell is rigid (undeformable) then $\upsilon = np_n\mathbf{V} = (\mathbf{V}, \mathbf{n})$ and (1.2) takes the form

$$\Delta p = \beta(\mathbf{V}, \mathbf{n}) \tag{1.3}$$

In carrying out specific calculations, it is convenient to parametrize the shell using the Cartesian coordinates $x^1 = x$, $x^2 = y$ and to describe the surface by means of an explicit expression $z = f(x, y)$. Then

$$\mathbf{r} = \{x, y, z = f(x, y)\}, \quad \mathbf{r}_1 = \{1, 0, f_x\}$$

$$\mathbf{r}_2 = \{0, 1, f_y\}, \quad \mathbf{n}^0 = \{-f_x, -f_y, 1\}, \quad \mathbf{n} = \mathbf{n}^0 / |\mathbf{n}^0|$$

We substitute this into (1.3) and take account of the fact that, in the case of a hollow shell, the squares of the derivatives f_x, f_y may be neglected compared with unity. Finally, we obtain

$$\Delta p = -\beta(f_x \upsilon_x + f_y \upsilon_y - \upsilon_z) \tag{1.4}$$

Comparatively simple formulae for calculating the integral characteristics can be obtained in Cartesian coordinates. For example, in the case of the z-axis projection of the principal vector of the aerodynamic interaction forces, we find

$$P = \iint_\sigma \Delta p(x^1, x^2) \cos\gamma d\sigma = \iint_S \Delta p(x, y) dx dy$$

Here, σ is the supporting surface and S is its projection on the x, y-plane.

2. Let us now assume that the shell is deformed. In order to calculate the excess pressure in this case, it is necessary to determine the position of the normal to the deformed surface. We will use the notation $\mathbf{U} = u\mathbf{r}_1 + \upsilon\mathbf{r}_2 + w\mathbf{n}$, where $\mathbf{U}$ is the displacement vector of the points of the median surface. The deformed surface will then be defined by the radius-vector $\mathbf{R} = \mathbf{r} + \mathbf{U}$. Using the notation $\mathbf{R}_k = \partial \mathbf{R} / \partial x^k$ $(k = 1, 2)$, for the new position of the normal we shall have

$$\mathbf{n}' = [\mathbf{R}_1 \times \mathbf{R}_2] / |[\mathbf{R}_1 \times \mathbf{R}_2]| \tag{2.1}$$

Let us calculate the derivatives $\mathbf{R}_k$. Using the rules for the differentiation of vectors $\mathbf{r}_k$ and of a normal $\mathbf{n}$ [9], we obtain as a result

$$\begin{aligned}
&\mathbf{R}_k = A_{k1}\mathbf{r}_1 + A_{k2}\mathbf{r}_2 + B_k\mathbf{n}\\
&A_{k1} = \partial u/\partial x^k + C_{k1},\quad A_{k2} = \partial \upsilon/\partial x^k + C_{k2}\\
&B_k = \partial w/\partial x^k + b_{k1}u + b_{k2}\upsilon,\quad C_{kl} = G_{k1}^l u + G_{k2}^l \upsilon + b_k^l w\\
&k,l = 1,2
\end{aligned}\tag{2.2}$$

Here, $b_{kl} = b_{ki}g^{il}$, G_{il}^k are second-order Cristoffel symbols and g^{il} are the contravariant component of the metric tensor

$$g^{11} = \frac{g_{22}}{g},\quad g^{22} = \frac{g_{11}}{g},\quad g^{12} = g^{21} = -\frac{g_{12}}{g},\quad g = g_{11}g_{22} - g_{12}^2.$$

Next, using (2.2), we calculate $[\mathbf{R}_1 \times \mathbf{R}_2]$ and use the formulae

$$[\mathbf{r}_1 \times \mathbf{r}_2] = \sqrt{g}\mathbf{n},\quad \sqrt{g}[\mathbf{r}_k \times \mathbf{n}] = g_{k2}\mathbf{r}_1 - g_{k1}\mathbf{r}_2,\quad \|[\mathbf{R}_1 \times \mathbf{R}_2]\| = [(\mathbf{R}_1,\mathbf{R}_1)(\mathbf{R}_2,\mathbf{R}_2) - (\mathbf{R}_1,\mathbf{R}_2)^2]^{1/2}.$$

All of the resulting expressions are introduced into (2.2) and (2.1) and the final result is simplified. On the basis of the hypotheses of the linear theory of shells, we obtain

$$\mathbf{n}' = \mathbf{n} - A_1\mathbf{r}_1 - A_2\mathbf{r}_2,\quad A_k = g^{k1}B_1 + g^{k2}B_2,\quad k = 1,2 \tag{2.3}$$

Note that an expansion of the displacement vector $\mathbf{U}$ over the basis set unit vectors has to be introduced into the computational formulae. Since $\mathbf{r}_k = |\mathbf{r}_k|\mathbf{e}_k = \sqrt{(g_{kk})}\mathbf{e}_k$, we then obtain $\mathbf{U} = u\sqrt{(g_{11})}\mathbf{e}_1 + \upsilon\sqrt{(g_{22})}\mathbf{e}_2 + w\mathbf{n}$ and, here, $u^* = u\sqrt{(g_{11})}$ and $\upsilon^* = \upsilon\sqrt{(g_{22})}$ will be the physical components of the displacement vector. Having expressed u and υ using this, we write the expressions for B_k

$$B_k = \frac{\partial w}{\partial x^k} + \frac{b_{k1}}{\sqrt{g_{11}}}u^* + \frac{b_{k2}}{\sqrt{g_{22}}}\upsilon^*$$

We will now present the computational formulae. For a hollow shell $|f_x| \ll 1$, $|f_y| \ll 1$ and, hence, $g^{kk} \cong g_{kk} \cong 1$, $|g^{12}| \cong |g_{12}| \ll 1$, $|g| \cong 1$ which implies that $A_k \cong B_k$, $|A_1 f_x + A_2 f_y| \ll 1$. The coefficients $b_{11} = f_{xx}/\sqrt{g}$, $b_{22} = f_{yy}/\sqrt{g}$, $b_{12} = f_{xy}/\sqrt{g}$ and these are quantities of the order of the principal curvatures and torsions and therefore $B_k \cong \partial w/\partial x^k$. Taking account of these estimates from (2.3), we obtain

$$\mathbf{n}' \cong \left\{-\left(f_x + \frac{\partial w}{\partial x}\right), -\left(f_y + \frac{\partial w}{\partial y}\right), 1\right\}$$

$$(\mathbf{V}',\mathbf{n}') = -\left(f_x + \frac{\partial w}{\partial x}\right)\upsilon_x - \left(f_y + \frac{\partial w}{\partial y}\right) + \upsilon_z \tag{2.4}$$

The complete expression for $\upsilon(t)$ is defined as the sum: $\upsilon(t) = \partial w/\partial t + (\mathbf{V},\ \mathbf{n}')$ and, substituting this together with (2.4) into (1.3), we obtain an expression for the excess pressure

$$\Delta p = \beta\left[\frac{\partial w}{\partial t} - \left(f_x + \frac{\partial w}{\partial x}\right)\upsilon_x - \left(f_y + \frac{\partial w}{\partial y}\right)\upsilon_y + \upsilon_z\right] \tag{2.5}$$

The equations of the linear theory of hollow shells have the form (see [5], for example)

$$D\Delta^2 w - hL(\Phi) - q = 0, \quad \Delta^2\Phi + EL(w) = 0, \quad L(f) = k_x \frac{\partial^2 f}{\partial x^2} + k_y \frac{\partial^2 f}{\partial y^2} \tag{2.6}$$

Here Δ is the Laplace operator, Φ is a function of the stresses, k_x and k_y are the principal curvatures of the shell, h is its thickness and E is Young's modulus of the material. The transverse load q is made up of the inertial forces and the excess pressure: $q = -\Delta p - \rho h\partial^2 w/\partial t^2$. On substituting this into (2.6), we obtain a system on the basis of which the oscillations and dynamical stability of a hollow shell can be investigated. System (2.6) is closed by the addition of the initial and boundary conditions to it.

Let us separate out the "static" solution $w_0(x, y), \Phi(x, y: \Phi_0(x, y)$ in (2.6)

$$D\Delta^2 w_0 - L(\Phi_0) - \beta(\mathbf{v}, \operatorname{grad} w_0) = q_0$$

$$\Delta^2\Phi_0 + EL(w_0) = -0, \quad \mathbf{v} = \{v_x, v_y\}, \quad q_0 = \beta[(\mathbf{v}, \operatorname{grad} f) + v_z]$$

which satisfies the boundary conditions of the problem. Here, we assume that the shell does not lose stability under the action of a static load q_0. For the "dynamic" deflection $W(x, y, t)$ and the stress function $\Phi_1(x, y, t)$, we obtain after this a homogeneous system with the same boundary conditions and specified initial conditions

$$D\Delta^2 W - hL(\Phi_1) + \rho h\frac{\Delta^2 W}{\partial t^2} + \beta\frac{\partial W}{\partial t} - \beta(\mathbf{v}, gradW) = 0$$

$$\Delta^2\Phi_1 + EL(W) = 0 \tag{2.7}$$

On account of the problem of stability, we shall investigate the oscillations of the shell in the class of functions

$$W = \varphi(x, y)e^{\omega t}, \quad \Phi_1 = F(x, y)\, e^{\omega t}$$

On substituting these expressions into system (2.7), we obtain

$$D\Delta^2\varphi - hL(F) - \beta(\mathbf{v}, \operatorname{grad}\varphi) - \lambda\varphi = 0$$

$$\Delta^2 F + EL(\varphi) = 0, \quad \lambda = -\rho h\omega^2 - \beta\omega \tag{2.8}$$

Together with the boundary conditions, system (2.8) constitutes of problem from which the eigenvalues λ and the eigenfunctions φ, F must be found.

Let $\lambda_1 = \alpha_1 + \beta_1 i$ be the first eigenvalue. Oscillations with frequencies with a negative real part will be stable. On the basis of the equality $\rho h\omega^2 + \beta\omega + \alpha_1 + \beta_1 i = 0$, the inequality $\alpha_1\beta^2 > \rho h\beta_1^2$ corresponds to the condition Re $\omega < 0$. Since α_1 and β_1 depend on v_x, v_y, the condition which has been written out defines a curve in the plane of the components of the velocity vector v_x, v_y which separates the domains of stable and unstable oscillations.

Let us make system (2.8) more specific by considering several examples. *A rectangular plate.* In this case $k_x = k_y = 0$, $F = 0$ and it follows from the first equation of (2.8) that

$$D\Delta^2\varphi - \beta(\mathbf{v} \operatorname{grad}\varphi) - \lambda\varphi = 0 \tag{2.9}$$

Solutions of certain problems based on this equation were obtained in [12] and revealed new mechanical effects.

A circular plate. If the boundary conditions are the same over the whole contour, it is natural to put $v_y = 0$ and to write (2.9) in polar coordinates

$$D\Delta^2\varphi - \beta\left(\frac{\partial\varphi}{\partial r}\cos\theta - \frac{\partial\varphi}{r d\theta}\sin\theta\right)\upsilon_x - \lambda\varphi = 0 \tag{2.10}$$

A cylindrical panel which is rectangular in the plan view. The x-axis is directed along the generatrix. Then, $k_x = 0$, $k_y = R^{-1}$ and, from (2.8), we have

$$D\Delta^2\varphi - \frac{h}{R}\frac{\partial^2 F}{\partial x^2} - \beta(\mathbf{v}, \operatorname{grad}\varphi) - \lambda\varphi = 0$$

$$\Delta^2 F + \frac{E}{R}\frac{\partial^2\varphi}{\partial x^2} = 0$$

A spherical panel which is rectangular in the plan view. Since $k_x = k_y = R^{-1}$, it follows from (2.8) that

$$D\Delta^2\varphi - \frac{h}{R}\Delta F - \beta(\mathbf{v}, \operatorname{grad}\varphi) - \lambda\varphi = 0$$

$$\Delta^2 F + \frac{E}{R}\Delta\varphi = 0$$

Circular spherical panel. If the boundary conditions are the same over the entire contour, we then put $\upsilon_y = 0$ and, in polar coordinates, we obtain

$$\begin{aligned} &D\Delta^2\varphi - \frac{h}{R}\Delta F - \beta\left(\frac{\partial\varphi}{\partial r}\cos\theta - \frac{\partial\varphi}{r\partial\theta}\sin\theta\right)\upsilon_x - \lambda\varphi = 0 \\ &\Delta^2 F + \frac{E}{R}\Delta\varphi = 0 \end{aligned} \tag{2.11}$$

In formulae (2.10) and (2.11), the Laplace operator has to be written in polar coordinates.

REFERENCES

1. MOVCHAN A. A., On the oscillations of a plate which is moving in a gas. *Prikl. Mat. Mekh.* **20**, 2, 211–222, 1956.
2. MOVCHAN A. A., On the stability of a panel which is moving in a gas. *Prikl. Mat. Mekh.* **21**, 2, 231–243, 1957.
3. STEPANOV R. D., On the flutter of cylindrical shells and panels moving in a gas flow. *Prikl. Mat. Mekh.* **21**, 5, 644–657, 1957.
4. BOLOTIN V. V., *Non-conservative Problems in the Theory of Elastic Stability*. Fizmatgiz, Moscow, 1961.
5. OGIBALOV P. M., *Questions of the Dynamics and Stability of Shells*. Izd. Mosk. Gos. Univ, Moscow, 1963.
6. MATYASH V. I., Flutter of a viscoelastic plate. *Mekh. Polimerov* **6**, 1077–1083, 1971.
7. LARIONOV G. S., Non-linear flutter of a viscoelastic plate. *Izd. Akad. Nauk SSSR*, MTT **4**, 95–100, 1974.
8. OGIBALOV P. M., LOMAKIN V. V. and KISHKIN B. P., *Mechanics of Polymers*. Izd. Mosk. Gos. Univ., Moscow, 1975.
9. RASHEVSKII P. K., *Course of Differential Geometry*. Gostekhizdat, Moscow, 1950.
10. IL'YUSHIN A. A., The law of plane sections in the aerodynamics of large supersonic velocities. *Prikl. Mat. Mekh.* **20**, 6, 733–755, 1956.
11. IL'YUSHIN A. A., *Mechanics of a Continuous Medium*. Izd. Mosk. Gos. Univ., Moscow, 1990.

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